3D Image Reconstruction from Exponential X-ray Projections: a Completeness Condition and an Inversion Formula

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Abstract— **This work concerns the problem of reconstructing a 3D image from exponential X-ray (parallel-beam) projections. It is shown that exact reconstruction can be achieved when the projections are known on a set of directions Ω which satisfies the Orlov condition for non-attenuated projections. More specifically, it is shown that exact reconstruction can be achieved when the set Ω is intersected by every great circle on the unit sphere, provided the product μR is sufficiently small, where R is the radius of the region where the image is non-zero and μ is the attenuation coefficient. A reconstruction method is suggested and simulation results are provided to demonstrate the exactness and usefulness of the method.**

Keywords— **SPECT, attenuation correction, exponential parallel-beam projections, Orlov's condition**

I. Introduction

This work concerns the mathematical problem of reconstructing a 3D image $f(\underline{x})$ from exponential X-ray (parallel-beam) projections. This problem is stated as follows:
Let $\underline{\theta}$ be some unit vector and let

$$p(\underline{\theta},u,v) = \int_{-\infty}^{\infty} dt\, f(u\,\underline{\alpha} + v\,\underline{\beta} + t\,\underline{\theta})\,\exp(\mu t) \qquad (u,v) \in \mathbb{R}^2 \tag{1}$$

be the exponential X-ray projection of $f(\underline{x})$ in the direction $\underline{\theta}$. Given $p(\underline{\theta},\cdot,\cdot)$ for all vectors $\underline{\theta}$ in a subset Ω of the unit sphere, determine f.

Note that the vectors $\underline{\alpha}$ and $\underline{\beta}$ in (1) are unit orthogonal vectors perpendicular to $\underline{\theta}$, while u and v are Cartesian coordinates used to specify different lines in the direction $\underline{\theta}$. The constant μ is the attenuation coefficient.

In the case where $\mu = 0$, $p(\underline{\theta},u,v)$ is a non-attenuated parallel-beam projection of f. The reconstruction theory for this situation has been widely covered in the literature. Of particular interest is the work of Orlov [1] who showed that exact reconstruction of $f(\underline{x})$ is possible when Ω is intersected by every great circle on the unit sphere.

When $\mu \neq 0$, reconstruction of f from $p(\underline{\theta},u,v)$ is more difficult and only particular geometries have been investigated [2]-[9]. From these works, it is known that exact reconstruction can be achieved when Ω is anyone of the following sets: a great circle [2,3], the full sphere [4,5], a union of great circles [6], a semi great circle [7], a semi equatorial band [8] and a union of small circles satisfying Orlov's condition (RSH-SPECT geometry) [9].

In this work, we show that Orlov's condition is actually valid for reconstruction of $f(\underline{x})$ from $p(\underline{\theta},u,v)$ when $\mu \neq 0$. More specifically, we show that exact reconstruction of $f(\underline{x})$ is possible when Ω is intersected by every great circle on the unit sphere provided the product μR is sufficiently small, where R is the radius of the region where $f(\underline{x})$ is non-zero.

Care must be taken with the definition of Ω in the case where $\mu \neq 0$. In the non-attenuated case where $\mu = 0$, $p(\underline{\theta},u,v) = p(-\underline{\theta},u,v)$ and it is customary to assume that Ω is symmetric (i.e. if $\underline{\theta} \in \Omega$, then $-\underline{\theta} \in \Omega$). When $\mu \neq 0$, the reconstruction problem is different for symmetric and non-symmetric sets Ω because $p(\underline{\theta},u,v) \neq p(-\underline{\theta},u,v)$. The results in this paper apply to both types of sets.

This work finds its main application in SPECT imaging. Indeed, assuming that the attenuation is constant in the activity region, it is known that ideal SPECT data are related to exponential X-ray projections by a set of multiplicative weights defined from the attenuation map [10].

II. Method

Suppose that $\mu = 0$. In this case, it is known that there exists a filter $h(\underline{\theta},u,v)$ such that

$$f(\underline{x}) = \int_\Omega d\underline{\theta}\,(h * p_{\mu=0})(\underline{\theta}, u = \underline{x}\cdot\underline{\alpha}, v = \underline{x}\cdot\underline{\beta})\,, \tag{2}$$

where the symbol $*$ denotes a convolution operation.

Our inversion formula is based on the existence of the above filter $h(\underline{\theta},u,v)$ and the relation

$$f(\underline{x}) = f_0(\underline{x}) + (W * f)(\underline{x}) \tag{3}$$

where

$$f_0(\underline{x}) = \int_\Omega d\underline{\theta}\,\exp(-\mu\,\underline{x}\cdot\underline{\theta})\,(h*p)(\underline{\theta},\underline{x}\cdot\underline{\alpha},\underline{x}\cdot\underline{\beta})\,, \tag{4}$$

and

$$W(\underline{x}) = \int_\Omega d\underline{\theta}\,(1 - \exp(-\mu\,\underline{x}\cdot\underline{\theta}))\,h(\underline{\theta},\underline{x}\cdot\underline{\alpha},\underline{x}\cdot\underline{\beta})\,. \tag{5}$$

Let R be such that $f(\underline{x}) = 0$ for $|\underline{x}| > R$ and let

$$\chi(\underline{x}) = \begin{cases} 1 & \text{if } |\underline{x}| < R \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

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In practice R is always finite since f is physically restricted to a finite region. Using χ, the integral equation (3) can be rewritten in the form

$$f = \chi f_0 + \chi (W * f) = \chi f_0 + K f \tag{7}$$

where K is an operator such that $K f = \chi(W * f)$. Because $W(\underline{x})$ tends to zero when μ tends to zero and $\chi(\underline{x})$ restricts the action of W to the region $|\underline{x}| < R$, $||K|| < 1$ for μR sufficiently small and equation (7) admits then the unique solution

$$f = f_0 + \sum_{l=1}^{\infty} \hat{K}^l \chi f_0 \,. \tag{8}$$

III. Simulation

We have implemented formula (8) for three different trajectories: a great circle, a semi-great circle, and the RSH-SPECT geometry with a slant angle of 30° and three detector positions separated by 60°. Figure 1 shows the results obtained from computer-simulated projections of a heart phantom made up of ellipsoids, with 20% activity in the ventricules. The attenuation coefficient was $\mu = 0.0152/\text{mm}$ and the radius of the field-of-view was $R = 74$ mm. For each geometry, there were 150 projections (each of 128×128 square pixels of side 1.2 mm) and the reconstruction was performed on a grid of 128^3 cubic voxels of side 1.2 mm. The quality of the results demonstrates the validity of the method for practical data parameters.

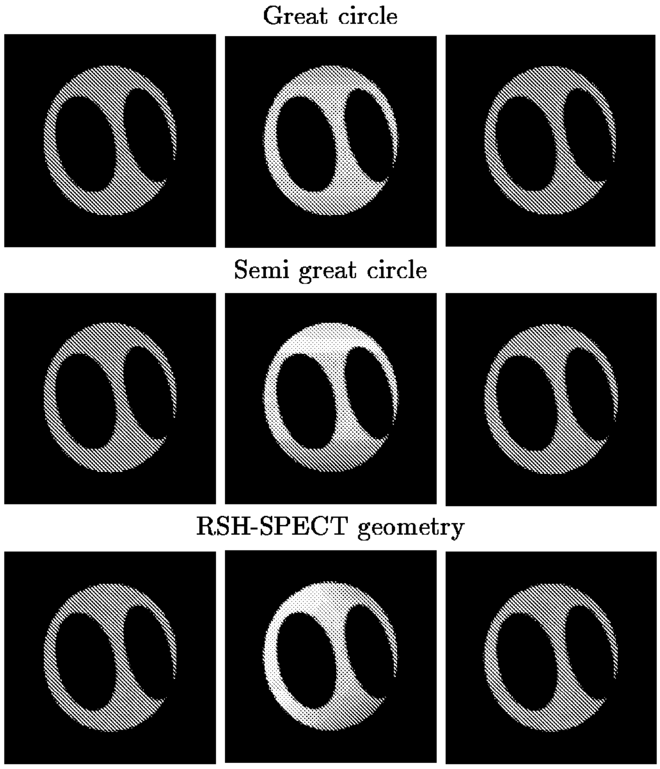

Fig. 1. Left column: reconstruction from non-attenuated data using (2). Middle column: reconstruction $f_0(\underline{x})$ (equation (4)). Right column: implementation of formula (8) with 4 terms. The images are displayed using the gray-scale [0.8, 1.2] window centered on the heart-wall activity value of 1. The average relative error for the reconstructions in the right column is approximately 3%. Reconstruction time: about 5 min cpu per iteration on a SUN ULTRA 10.

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