Probabilistic Framework for the Characterization of Surfaces and Edges in Range Images, with Application to Edge Detection: Supplemental material

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Abstract

In this document, we provide some additional mathematical developments regarding some of the expressions obtained in the main paper as well as some additional results.

1 Experimental estimation of the pdfs of orientation and of a single range

In Figure 1, we estimated the pdf $f_{\Theta}(\theta)$ of orientations and the pdf $f_{Z_{\mathbf{p}}}(z_{\mathbf{p}})$ for additional datasets:

- NYU Depth Dataset [3] (1449 images captured by a Kinect 1),
- MPI Sintel Depth Training Data [2] (1064 synthetic images),
- RGB-D SLAM Dataset [5]: handheld and object reconstruction category, and
- Our custom dataset for jump edge evaluation (CamCube images).

The histograms were obtained by sampling the images every eight pixels in order to reduce the effect of the noise when estimating the orientation θ.

We note that whether the experimental angle distribution on a given dataset is a uniform pdf or not will not necessarily validate or invalidate our physical assumption. To the contrary, if the assumption is not met for a given dataset, it just tells us that the camera is not capable to capture some surface angles or that the dataset is not representative of all the use cases.

2 Determination of the analytic expressions of $f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}}|\bar{z}_{\mathbf{q}}, \mathcal{S}_{\mathbf{pq}})$ and $f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}})$ for bounded values of $z_{\mathbf{p}} \in [z_{min}, z_{max}]$

In this section, we determine the analytic expression of pdf of $f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}}|\bar{z}_{\mathbf{q}}, \mathcal{S}_{\mathbf{pq}})$ and $f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}})$ when we consider that the range values are bounded to $z_{\mathbf{p}} \in [z_{min}, z_{max}]$.

First, we note that the Reciprocal pdf of $f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}})$ obtained in the article is an approximation and cannot be defined for unbounded ranges. Indeed, we need to have bounded values to have the integral of $f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}})$ be finite. However, to derive its exact expression, we should also bound the values of $\bar{z}_{\mathbf{p}}$ in the pdf $f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}}|\bar{z}_{\mathbf{q}}, \mathcal{S}_{\mathbf{pq}})$. The expression of the pdf becomes

$$f_{\bar{Z}_{\mathbf{p}}}(\bar{z}_{\mathbf{p}}|\bar{z}_{\mathbf{q}}, \mathcal{S}_{\mathbf{pq}}) = \begin{cases} \frac{1}{C_{\mathbf{qp}}\bar{z}_{\mathbf{q}}s_{\mathbf{qp}}\left(1+\left[\frac{\bar{z}_{\mathbf{p}}-\bar{z}_{\mathbf{q}}l_{\mathbf{qp}}}{\bar{z}_{\mathbf{q}}s_{\mathbf{qp}}}\right]^2\right)} & \bar{z}_{\mathbf{p}} \in [z_{min}, z_{max}] \\ 0 & \text{otherwise,} \end{cases}$$

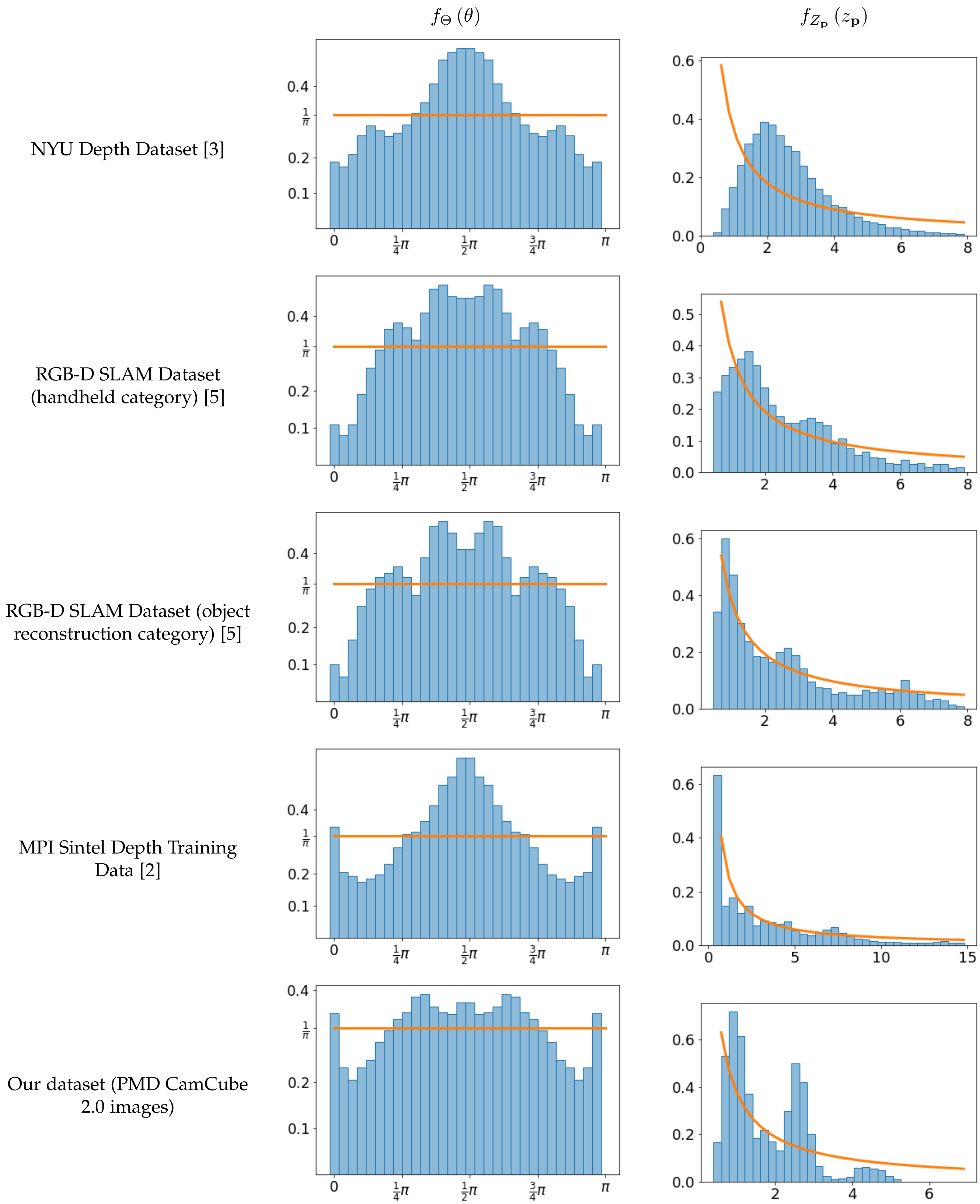

Figure 1. Estimation of $f_\Theta(\theta)$ and $f_{Z_\mathbf{p}}(z_\mathbf{p})$ by the histograms for several datasets.

where $C_{\mathbf{qp}}$ is a normalization constant such that the integral of $f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}|\bar{z}_{\mathbf{q}}, \mathcal{S}_{\mathbf{pq}}\right)$ on $[z_{min}, z_{max}]$ is 1:

$$C_{\mathbf{qp}} = \arctan\left(\frac{z_{max} - \bar{z}_{\mathbf{q}} l_{\mathbf{qp}}}{\bar{z}_{\mathbf{q}} s_{\mathbf{qp}}}\right) - \arctan\left(\frac{z_{min} - \bar{z}_{\mathbf{q}} l_{\mathbf{qp}}}{\bar{z}_{\mathbf{q}} s_{\mathbf{qp}}}\right).$$

We now have

$$\frac{f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}|\bar{z}_{\mathbf{q}}, \mathcal{S}_{\mathbf{pq}}\right)}{f_{\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{q}}|\bar{z}_{\mathbf{p}}, \mathcal{S}_{\mathbf{pq}}\right)} = \frac{f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}\right)}{f_{\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{q}}\right)} = \frac{C_{\mathbf{pq}}}{C_{\mathbf{qp}}}\frac{\bar{z}_{\mathbf{q}}}{\bar{z}_{\mathbf{p}}},$$

where $C_{\mathbf{qp}}$ and $C_{\mathbf{pq}}$ depend only on $\bar{z}_{\mathbf{q}}$ and $\bar{z}_{\mathbf{p}}$, respectively. Thus, the pdf $f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}\right)$ must have an expression of the form

$$f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}\right) = \frac{C_{\mathbf{pq}}}{K_{\mathbf{pq}}\bar{z}_{\mathbf{p}}},$$

where $K_{\mathbf{pq}}$ is a normalizing constant. We note that this expression can also be verified by supposing that $f_{\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{q}}\right) = \frac{C_{\mathbf{qp}}}{K_{\mathbf{qp}}\bar{z}_{\mathbf{q}}}$ and calculating the integral

$$\begin{aligned} f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}\right) &= \int_{z_{min}}^{z_{max}} f_{\bar{Z}_{\mathbf{p}},\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{p}}, \bar{z}_{\mathbf{q}}\right) d\bar{z}_{\mathbf{q}} \\ &= \int_{z_{min}}^{z_{max}} f_{\bar{Z}_{\mathbf{p}},\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{p}}, \bar{z}_{\mathbf{q}}|\mathcal{J}_{\mathbf{pq}}\right)\pi_{\mathcal{J}} d\bar{z}_{\mathbf{q}} + \int_{z_{min}}^{z_{max}} f_{\bar{Z}_{\mathbf{p}},\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{p}}, \bar{z}_{\mathbf{q}}|\mathcal{S}_{\mathbf{pq}}\right)\left(1-\pi_{\mathcal{J}}\right) d\bar{z}_{\mathbf{q}} \\ &= \pi_{\mathcal{J}} f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}\right) + \left(1-\pi_{\mathcal{J}}\right)\int_{z_{min}}^{z_{max}} f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}|\bar{z}_{\mathbf{q}}, \mathcal{S}_{\mathbf{pq}}\right) f_{\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{q}}\right) d\bar{z}_{\mathbf{q}}. \end{aligned}$$

To obtain the Reciprocal pdf approximation, we need to remark that $C_{\mathbf{pq}} \approx 1$ everywhere expect for values of $\bar{z}_{\mathbf{p}}$ extremely close to z_{min} or z_{max}. Thus, almost everywhere, we have

$$f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}\right) \approx \frac{1}{K\bar{z}_{\mathbf{p}}},$$

with $K = \log z_{max} - \log z_{min}$.

3 Determination of $f\left(\boldsymbol{z}^{-1}|\mathcal{S}_{\mathbf{p}_1\mathbf{p}_N}\right)$

In this section, we determine the expression of $f\left(\boldsymbol{z}^{-1}|\mathcal{S}_{\mathbf{p}_1\mathbf{p}_N}\right)$ given in Equation (15) of the paper.

Given two pixels $\mathbf{p}_i$ and $\mathbf{p}_j$ and their noiseless ranges $\bar{z}_i$ and $\bar{z}_j$, and assuming that all these points belong to a same planar surface, the range $\bar{z}_k$ of any pixel $\mathbf{p}_k$, such that $\mathbf{p}_k = \mathbf{p}_i + \alpha_{ijk}\left(\mathbf{p}_j - \mathbf{p}_i\right)$, is related to the other two ranges by

$$\bar{z}_k^{-1} = \bar{z}_i^{-1} + \mathrm{a}_{ijk}\left(\bar{z}_j^{-1} - \bar{z}_i^{-1}\right). \tag{1}$$

For N ranges, we have the following relation in matrix form:

$$\bar{\boldsymbol{z}}^{-1} = \mathrm{A}\bar{\boldsymbol{\beta}}, \tag{2}$$

where $\bar{\boldsymbol{z}}^{-1} = \left(\bar{z}_1^{-1} \cdots \bar{z}_i^{-1} \cdots \bar{z}_N^{-1}\right)^T$, $\bar{\boldsymbol{\beta}} = \left(\bar{z}_1^{-1}\ \bar{z}_N^{-1}\right)^T$, and A is a $N \times 2$ matrix defined as

$$\mathrm{A} = \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 - \mathrm{a}_{1Ni} & \mathrm{a}_{1Ni} \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Noting that $Z_i^{-1} = \left(\bar{Z}_i + N_i\right)^{-1}$ and considering that the standard deviation σ_i of the noise relative to z_i is much smaller than z_i ($\sigma_i \ll z_i$), we can approximate Z_i^{-1} as the noiseless inverse range $\bar{Z}_i^{-1}$ plus a Gaussian noise with a zero mean and a variance equal to $\left(\frac{\sigma_i}{z_i^2}\right)^2$. Likewise, the noise of the inverse measured ranges $\boldsymbol{z}^{-1} = \left(z_1^{-1} \cdots\ z_i^{-1} \cdots\ z_N^{-1}\right)^T$ can be approximated by a multivariate Gaussian random variable $\boldsymbol{N}$ with zero mean and whose variance matrix is

$$\boldsymbol{\Sigma} = \begin{pmatrix} \left[\frac{\sigma_1}{z_1^2}\right]^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \left[\frac{\sigma_N}{z_N^2}\right]^2 \end{pmatrix}, \tag{4}$$

i.e.

$$f_{\boldsymbol{N}}\left(\boldsymbol{n}\right) = (2\pi)^{-\frac{N+1}{2}} \left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\boldsymbol{n}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{n}\right], \tag{5}$$

where $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.

The RV $\boldsymbol{Z}^{-1}$ can be expressed as the sum of the noiseless inverse ranges $\bar{\boldsymbol{Z}}^{-1} = \mathrm{A}\bar{\boldsymbol{\beta}}$ and the noise RV $\boldsymbol{N}$

$$\boldsymbol{Z}^{-1} = \mathrm{A}\bar{\boldsymbol{\beta}} + \boldsymbol{N}.$$

Its pdf can be obtained by marginalizing over $\bar{\boldsymbol{\beta}}$:

$$f\left(\boldsymbol{z}^{-1}|\mathcal{S}_{\mathbf{p}_1\mathbf{p}_N}\right) = \int_{\bar{\boldsymbol{\beta}}} G\left(\boldsymbol{z}^{-1}; \mathrm{A}\bar{\boldsymbol{\beta}}, \boldsymbol{\Sigma}\right) f\left(\bar{\boldsymbol{\beta}}\right) d\bar{\boldsymbol{\beta}}, \tag{6}$$

where $G\left(\boldsymbol{z}^{-1}; \mathrm{A}\bar{\boldsymbol{\beta}}, \boldsymbol{\Sigma}\right)$ is the multivariate Gaussian function with mean $\mathrm{A}\bar{\boldsymbol{\beta}}$ and variance $\boldsymbol{\Sigma}$. It is given by

$$G\left(\boldsymbol{z}^{-1}; \mathrm{A}\bar{\boldsymbol{\beta}}, \boldsymbol{\Sigma}\right) = (2\pi)^{-\frac{N+1}{2}} \left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\boldsymbol{z}^{-1} - \mathrm{A}\bar{\boldsymbol{\beta}}\right)^T \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{z}^{-1} - \mathrm{A}\bar{\boldsymbol{\beta}}\right)\right]. \tag{7}$$

The second term in the integral, $f\left(\bar{\boldsymbol{\beta}}\right) = f\left(\bar{z}_1^{-1}, \bar{z}_N^{-1}\right)$, is the joint pdf of two inverse noiseless ranges. In Section 3, we provide the expressions of the pdfs of the inverse ranges. This yields the product of a Cauchy pdf and a reciprocal pdf as given by

$$f\left(\bar{\boldsymbol{\beta}}\right) = \frac{C\left(\bar{z}_1^{-1}; \bar{z}_N^{-1} l_{\mathbf{p}_1\mathbf{p}_N}, \bar{z}_1^{-1} s_{\mathbf{p}_1\mathbf{p}_N}\right)}{\left[\ln z_{max} - \ln z_{min}\right] \bar{z}_N^{-1}}. \tag{8}$$

$\bar{\boldsymbol{\beta}}$ plays a particular role in the integral. We isolate the term with $\bar{\boldsymbol{\beta}}$ in the sum of squares in the Gaussian by noting that

$$\left(\boldsymbol{z}^{-1} - \mathrm{A}\bar{\boldsymbol{\beta}}\right)^T \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{z}^{-1} - \mathrm{A}\bar{\boldsymbol{\beta}}\right) = \left(\boldsymbol{z}^{-1} - \mathrm{A}\xi\right)^T \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{z}^{-1} - \mathrm{A}\xi\right) + \left(\bar{\boldsymbol{\beta}} - \xi\right)^T \left(\mathrm{A}^T\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)\left(\bar{\boldsymbol{\beta}} - \xi\right), \tag{9}$$

where $\xi = \begin{pmatrix} \widehat{z}_1^{-1} & \widehat{z}_N^{-1} \end{pmatrix}^T = \left(\mathrm{A}^T\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)^{-1}\mathrm{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}$; in fact, ξ is a 1×2 vector, which is the unbiased least squares estimator of $\bar{\boldsymbol{\beta}}$ [4]. Indeed, we have

$$\begin{aligned}
\left(\boldsymbol{z}^{-1} - \mathrm{A}\bar{\boldsymbol{\beta}}\right)^T \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{z}^{-1} - \mathrm{A}\bar{\boldsymbol{\beta}}\right) &= \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right)^T \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right) \\
&= \left\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right\|^2 \\
&= \left\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}} + \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right\|^2 \\
&= \left\|\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right) + \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right)\right\|^2 \\
&= \left\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right\|^2 + \left\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right\|^2 + \\
&\quad + 2\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^T \left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right) \\
&= \left\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1} - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right\|^2 + \left\|\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi - \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right\|^2 \\
&= \left(\boldsymbol{z}^{-1} - \mathrm{A}\xi\right)^T \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{z}^{-1} - \mathrm{A}\xi\right) + \left(\bar{\boldsymbol{\beta}} - \xi\right)^T \left(\mathrm{A}^T\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)\left(\bar{\boldsymbol{\beta}} - \xi\right),
\end{aligned}$$

since

$$\begin{aligned}
&\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right)\\
&=\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\right)\\
&=\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi-\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\\
&=\left[\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\right)\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)\right]^{T}\xi\\
&\quad-\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\\
&=\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\right)\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)^{-1}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}\right)^{T}\xi\\
&\quad-\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\\
&=\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}-\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}\right)^{T}\xi-\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\\
&=-\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\bar{\boldsymbol{\beta}}\\
&=-\left[\left(\bar{\boldsymbol{\beta}}^{T}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\right)\left(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)\right]^{T}\\
&=-\left(\bar{\boldsymbol{\beta}}^{T}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{z}^{-1}-\bar{\boldsymbol{\beta}}^{T}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{A}\xi\right)^{T}\\
&=-\left(\bar{\boldsymbol{\beta}}^{T}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}-\bar{\boldsymbol{\beta}}^{T}\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)^{-1}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}\right)^{T}\\
&=-\left(\bar{\boldsymbol{\beta}}^{T}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}-\bar{\boldsymbol{\beta}}^{T}\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{z}^{-1}\right)^{T}\\
&=0.
\end{aligned}$$

This leads to the decomposition of the Gaussian into

$$\begin{aligned}
G\left(\boldsymbol{z}^{-1};\mathrm{A}\bar{\boldsymbol{\beta}},\boldsymbol{\Sigma}\right)=&\left(2\pi\right)^{-\frac{N+1}{2}}\left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\boldsymbol{z}^{-1}-\mathrm{A}\xi\right)^{T}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{z}^{-1}-\mathrm{A}\xi\right)\right]\\
&\exp\left[-\frac{1}{2}\left(\bar{\boldsymbol{\beta}}-\xi\right)^{T}\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)\left(\bar{\boldsymbol{\beta}}-\xi\right)\right] \qquad (10)\\
=&G\left(\boldsymbol{z}^{-1};\mathrm{A}\xi,\boldsymbol{\Sigma}\right)\exp\left[-\frac{1}{2}\left(\bar{\boldsymbol{\beta}}-\xi\right)^{T}\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)\left(\bar{\boldsymbol{\beta}}-\xi\right)\right]\\
=&\left(2\pi\right)\left|\boldsymbol{\Sigma}_{\xi}\right|^{\frac{1}{2}}G\left(\boldsymbol{z}^{-1};\mathrm{A}\xi,\boldsymbol{\Sigma}\right)G\left(\xi;\bar{\boldsymbol{\beta}},\boldsymbol{\Sigma}_{\xi}\right) \qquad (11)
\end{aligned}$$

where $\boldsymbol{\Sigma}_{\xi}=\left(\mathrm{A}^{T}\boldsymbol{\Sigma}^{-1}\mathrm{A}\right)^{-1}$. The first Gaussian does not depend on $\bar{\boldsymbol{\beta}}$ and can be moved out of the integral. The second Gaussian can be decomposed into two univariate Gaussian [1] (formula 2.95 to 2.98 page 90)

$$G\left(\xi;\bar{\boldsymbol{\beta}},\boldsymbol{\Sigma}_{\xi}\right)=\left|\boldsymbol{\Sigma}_{\xi}\right|^{-\frac{1}{2}}\sqrt{[\boldsymbol{\Sigma}_{\xi}^{-1}]_{11}^{-1}[\boldsymbol{\Sigma}_{\xi}]_{22}}\,G\left(\bar{z}_{N}^{-1};\widehat{z}_{N}^{-1},\sqrt{[\boldsymbol{\Sigma}_{\xi}]_{22}}\right)G\left(\bar{z}_{1}^{-1};\widehat{z}_{1}^{-1}-[\boldsymbol{\Sigma}_{\xi}^{-1}]_{11}^{-1}[\boldsymbol{\Sigma}_{\xi}^{-1}]_{12}\left(\bar{z}_{N}^{-1}-\widehat{z}_{N}^{-1}\right),\sqrt{[\boldsymbol{\Sigma}_{\xi}^{-1}]_{11}^{-1}}\right) \qquad (12)$$

where $[M]_{ij}$ denotes the element at row i and column j of the matrix M. After we replace $G\left(\xi;\bar{\boldsymbol{\beta}},\boldsymbol{\Sigma}_{\xi}\right)$ and $f\left(\bar{\boldsymbol{\beta}}\right)$ by their modified expressions, we have an integral which is similar to the convolution obtained in Appendix A.5 and may thus be approximated by a Voigt function. Indeed, the joint pdf $f\left(z_{\mathbf{p}},z_{\mathbf{q}}\right)$ can be expressed as the marginalization over the noiseless ranges $\bar{z}_{\mathbf{p}}$ and $\bar{z}_{\mathbf{q}}$

$$
\begin{aligned}
f_{Z_{\mathbf{p}},Z_{\mathbf{q}}}\left(z_{\mathbf{p}},z_{\mathbf{q}}\right) &= \int_{\bar{z}_{\mathbf{p}}}\int_{\bar{z}_{\mathbf{q}}} f_{Z_{\mathbf{p}},Z_{\mathbf{q}}}\left(z_{\mathbf{p}},z_{\mathbf{q}}|\bar{z}_{\mathbf{p}},\bar{z}_{\mathbf{q}}\right) f_{\bar{Z}_{\mathbf{p}},\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{p}},\bar{z}_{\mathbf{q}}\right) d\bar{z}_{\mathbf{p}} d\bar{z}_{\mathbf{q}} \\
&= \int_{\bar{z}_{\mathbf{p}}}\int_{\bar{z}_{\mathbf{q}}} f_{Z_{\mathbf{p}}}\left(z_{\mathbf{p}}|\bar{z}_{\mathbf{p}}\right) f_{Z_{\mathbf{q}}}\left(z_{\mathbf{q}}|\bar{z}_{\mathbf{q}}\right) f_{\bar{Z}_{\mathbf{q}}}\left(\bar{z}_{\mathbf{q}}|\bar{z}_{\mathbf{p}}\right) f_{\bar{Z}_{\mathbf{p}}}\left(\bar{z}_{\mathbf{p}}\right) d\bar{z}_{\mathbf{p}} d\bar{z}_{\mathbf{q}} \\
&= \int_{\bar{z}_{\mathbf{p}}}\int_{\bar{z}_{\mathbf{q}}} G\left(z_{\mathbf{p}};\bar{z}_{\mathbf{p}},\sigma_{\mathbf{p}}\right) G\left(z_{\mathbf{q}};\bar{z}_{\mathbf{q}},\sigma_{\mathbf{q}}\right) C\left(\bar{z}_{\mathbf{q}};\bar{z}_{\mathbf{p}} l_{\mathbf{pq}},\bar{z}_{\mathbf{p}} s_{\mathbf{pq}}\right) \frac{1}{\left[\ln z_{max}-\ln z_{min}\right]\bar{z}_{\mathbf{p}}} d\bar{z}_{\mathbf{p}} d\bar{z}_{\mathbf{q}} \\
&= \int_{\bar{z}_{\mathbf{p}}} G\left(z_{\mathbf{p}};\bar{z}_{\mathbf{p}},\sigma_{\mathbf{p}}\right) V\left(z_{\mathbf{q}};\bar{z}_{\mathbf{p}} l_{\mathbf{pq}},\bar{z}_{\mathbf{p}} s_{\mathbf{pq}},\sigma_{\mathbf{q}}\right) \frac{1}{\left[\ln z_{max}-\ln z_{min}\right]\bar{z}_{\mathbf{p}}} d\bar{z}_{\mathbf{p}},
\end{aligned}
$$

and was found to be well approximated by

$$
f_{Z_{\mathbf{p}},Z_{\mathbf{q}}}\left(z_{\mathbf{p}},z_{\mathbf{q}}\right) \approx V\left(z_{\mathbf{q}}; z_{\mathbf{p}} l_{\mathbf{pq}}, z_{\mathbf{p}} s_{\mathbf{pq}}, \sqrt{\sigma_{\mathbf{p}}^2+\sigma_{\mathbf{q}}^2}\right) \frac{1}{\left[\ln z_{max}-\ln z_{min}\right] z_{\mathbf{p}}}.
$$

For the pdf $f\left(\boldsymbol{z}^{-1}|\mathcal{S}_{\mathbf{p}_1\mathbf{p}_N}\right)$, we have (condensing all the terms that do not depend on $\bar{\boldsymbol{\beta}}$ in the constant U)

$$
\begin{aligned}
f\left(\boldsymbol{z}^{-1}|\mathcal{S}_{\mathbf{p}_1\mathbf{p}_N}\right) =& U\int_{\bar{\boldsymbol{\beta}}} G\left(\bar{z}_N^{-1};\widehat{z}_N^{-1},\sqrt{[\boldsymbol{\Sigma}_\xi]_{22}}\right) G\left(\bar{z}_1^{-1};\widehat{z}_1^{-1}-[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}[\boldsymbol{\Sigma}_\xi^{-1}]_{12}\left(\bar{z}_N^{-1}-\widehat{z}_N^{-1}\right),\sqrt{[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}}\right) \\
& \frac{C\left(\bar{z}_1^{-1};\bar{z}_N^{-1} l_{\mathbf{p}_1\mathbf{p}_N},\bar{z}_1^{-1} s_{\mathbf{p}_1\mathbf{p}_N}\right)}{\left[\ln z_{max}-\ln z_{min}\right]\bar{z}_N^{-1}} d\bar{\boldsymbol{\beta}} \qquad (13) \\
=& U\int_{\bar{z}_N^{-1}} G\left(\bar{z}_N^{-1};\widehat{z}_N^{-1},\sqrt{[\boldsymbol{\Sigma}_\xi]_{22}}\right) \\
& V\left(\widehat{z}_1^{-1};[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}[\boldsymbol{\Sigma}_\xi^{-1}]_{12}\left(\bar{z}_N^{-1}-\widehat{z}_N^{-1}\right)+\bar{z}_N^{-1} l_{\mathbf{p}_1\mathbf{p}_N},\sqrt{[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}},\bar{z}_1^{-1} s_{\mathbf{p}_1\mathbf{p}_N}\right) \\
& \frac{1}{\left[\ln z_{max}-\ln z_{min}\right]\bar{z}_N^{-1}} d\bar{z}_N^{-1}. \qquad (14)
\end{aligned}
$$

Noting that $[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}[\boldsymbol{\Sigma}_\xi^{-1}]_{12}\left(\bar{z}_N^{-1}-\widehat{z}_N^{-1}\right)$ is much smaller than $\bar{z}_N^{-1} l_{\mathbf{p}_1\mathbf{p}_N}$ (because $\bar{z}_N^{-1}-\widehat{z}_N^{-1} \ll 1, [\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}[\boldsymbol{\Sigma}_\xi^{-1}]_{12} < 1, l_{\mathbf{p}_1\mathbf{p}_N} \approx 1$), we have

$$
f\left(\boldsymbol{z}^{-1}|\mathcal{S}_{\mathbf{p}_1\mathbf{p}_N}\right) \approx U\int_{\bar{z}_N^{-1}} G\left(\bar{z}_N^{-1};\widehat{z}_N^{-1},\sqrt{[\boldsymbol{\Sigma}_\xi]_{22}}\right) V\left(\widehat{z}_1^{-1};\bar{z}_N^{-1} l_{\mathbf{p}_1\mathbf{p}_N},\sqrt{[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}},\bar{z}_1^{-1} s_{\mathbf{p}_1\mathbf{p}_N}\right) \frac{1}{\left[\ln z_{max}-\ln z_{min}\right]\bar{z}_N^{-1}} d\bar{z}_N^{-1}, \tag{15}
$$

and finally,

$$
f\left(\boldsymbol{z}^{-1}|\mathcal{S}\right) \approx \frac{(2\pi)\sqrt{[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}[\boldsymbol{\Sigma}_\xi]_{22}}}{\left[\ln z_{max}-\ln z_{min}\right]\widehat{z}_N^{-1}} G\left(\boldsymbol{z}^{-1};\mathrm{A}\xi,\boldsymbol{\Sigma}\right) V\left(\widehat{z}_1^{-1};\widehat{z}_N^{-1} l_{1N},\sqrt{[\boldsymbol{\Sigma}_\xi^{-1}]_{11}^{-1}+[\boldsymbol{\Sigma}_\xi]_{22}},\widehat{z}_N^{-1} s_{1N}\right).
$$

4 Additional results and illustrations on our custom dataset

Additional results are presented in Figure 2, 3 and 4 for images of our custom dataset captured respectively by the Microsoft Kinect 1, the Microsoft Kinect 2 and the PMD CamCube 2.0.

References

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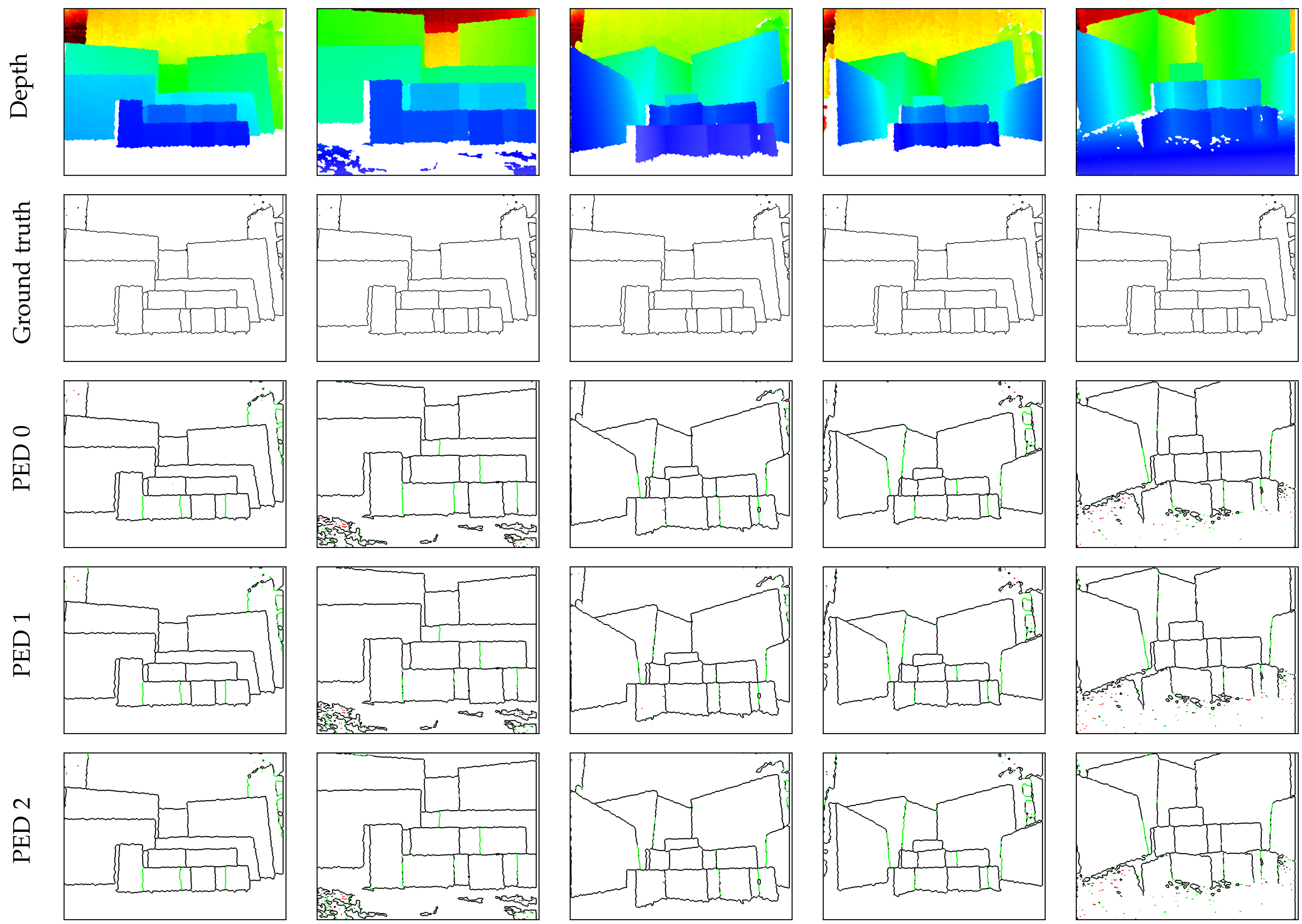

Figure 2. Edge detection results on Kinect 1 images of our custom dataset. The images were generated for the parameters maximizing the ODS score. Black, red and green pixels represent the true positives, false positives, and false negatives, respectively.

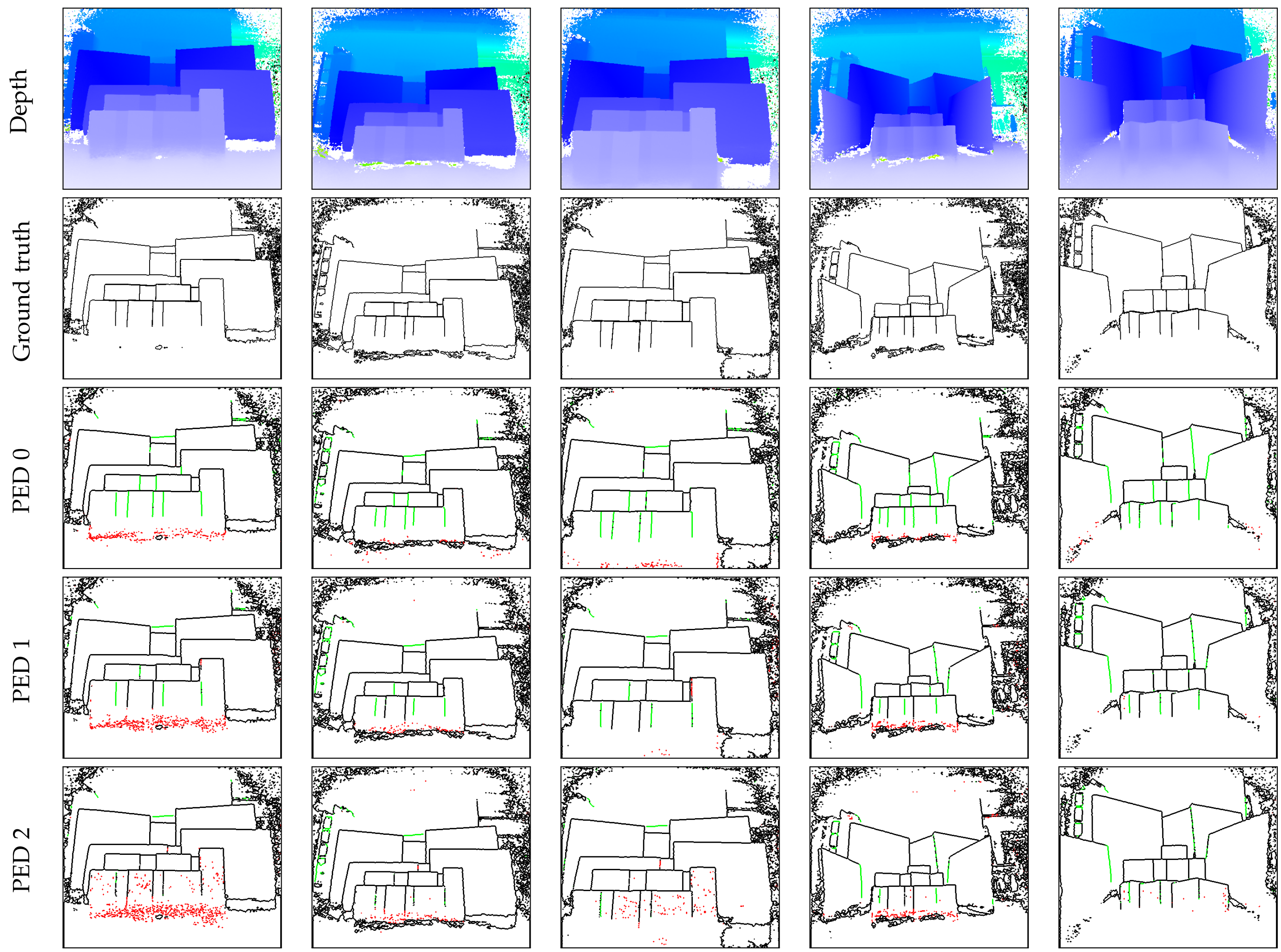

Figure 3. Edge detection results on Kinect 2 images of our custom dataset. The images were generated for the parameters maximizing the ODS score. Black, red and green pixels represent the true positives, false positives, and false negatives, respectively.

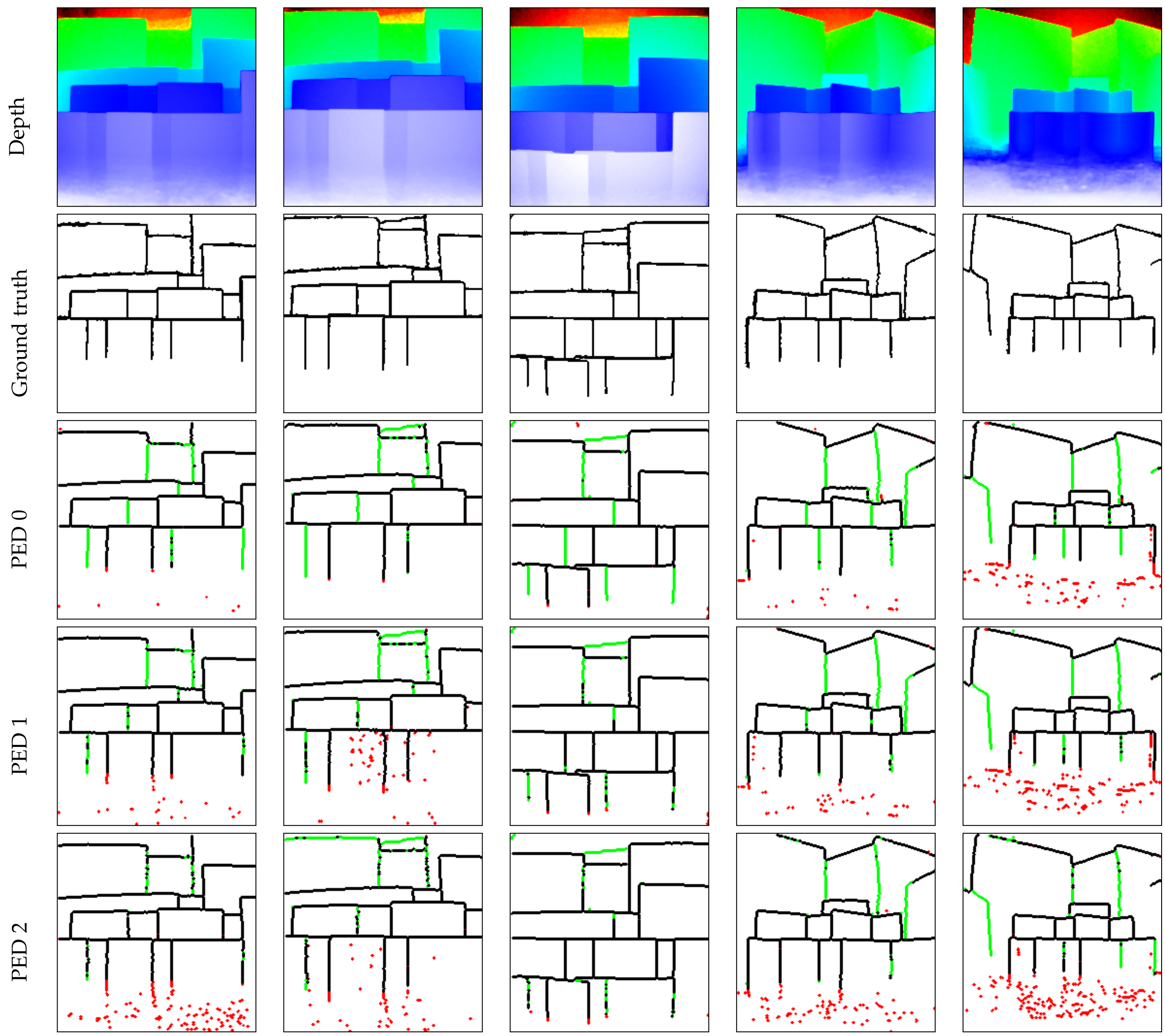

Figure 4. Edge detection results on PMD CamCube 2.0 images of our custom dataset. The images were generated for the parameters maximizing the ODS score. Black, red and green pixels represent the true positives, false positives, and false negatives, respectively.